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Investigation of the scattering of harmonic elastic waves by two collinear symmetric cracks using non-local theory

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Abstract. In this paper, the scattering of harmonic waves by two collinear symmetric cracks is studied by use of non-local theory. To overcome the mathematical difficulties, a one-dimensional non-local kernel is used instead of a two-dimensional one for the dynamic problem to obtain the stress occurring at the crack tips. The Fourier transform is applied and a mixed boundary-value problem is formulated. The solutions are obtained by means of the Schmidt method. This method is more exact and more appropriate than Eringen's for solving this kind of problem. Contrary to the classical elasticity solution, it is found that no stress singularity is present at the crack tip. The non-local dynamic elastic solutions yield a finite hoop stress near the crack tip, thus allowing for a fracture criterion based on the maximum dynamic stress hypothesis. The finite hoop stress at the crack tip depends on the crack length, the lattice parameter and the circular frequency of the incident wave.

Key words: cracks, harmonic waves scattering, non-local theory, Schmidt method, triple integral equations

1. Introduction

As is commonly known, one of the principal postulates of the traditional mechanics of continuous media is the principle of local action. This principle excludes action at a distance, and attributes changes occurring at a point of the medium to thermoenergetic factors acting at the point. Of necessity then, the classical theory, by restricting the response of the continuum to strictly local actions, constitutes a so-called local theory. A familiar example is provided by the conventional theory of elasticity, in which, when determining the stress at a point, one disregards the deformation and the temperature fields outside an arbitrarily small neighborhood at the point. However, the application of classical elasticity to micro-mechanics leads to some physically unreasonable results. A singularity appearing in a stress field is a typical one; the existence of stress singularities also results in difficulties in the development of experiments in micro-mechanics. In fact, the stress at the crack tip is finite. As a result of this, beginning with Griffith, all fracture criteria in practice today are based on other considerations, *e.g.* energy, and the J-integral [1].

In contrast to this local approach of zero-range internal interactions, modern non-local continuum mechanics, originated and developed in the last four decades, postulates that the local state at a point is influenced by the action of all particles of the body. This was done primarily by Edelen [2], Eringen [3], Green and Rivlin [4]. According to non-local theory, the stress at a point X in a body depends not only on the strain at point X but also on that at all other points of the body. This is different from classical theory. In classical theory, the stress at a point X in a body depends only on the strain at point X. In reference [5], the basic theory of non-local elasticity was stated with emphasis on the difference between the non-local theory

and classical continuum mechanics. The basic idea of non-local elasticity is to establish a relationship between macroscopic mechanical quantities and microscopic physical quantities within the framework of continuum mechanics. The constitutive theory of non-local elasticity has been developed widely [2], where the microstructures of the material have effect on the elastic modulus. It has been found that the microstructures of the material have their effect, not only on the constitutive equation, but also on the basic balance laws and boundary conditions [6–7].

Other advances have been made by the application of non-local elasticity to fields such as dislocation theory [8–9], solid defects [10–11] and fracture mechanics [12–13]. While the literature on the fundamental aspects of non-local continuum mechanics is relatively extensive, applications of the theory are not very numerous. The results, however, of those concrete problems that were solved display a rather remarkable agreement with experimental evidence. This can be used to predict the cohesive stress for various materials close to that obtained in atomic lattice dynamics [14–15]. Likewise, a non-local study of the secondary flow of viscous fluid in a pipe furnishes a streamline pattern similar to that obtained experimentally by Nikuradze [16]. Other examples of the effectiveness of the non-local approach are: (i) prediction of the dispersive character of elastic waves demonstrated experimentally (and lacking in the classical theory) [17] and (ii) calculation of the velocity of short Love waves whose non-local estimates agree better with seismological observations than the local ones [18]. Various of non-local theories have been formulated to address strain-gradient and size effects (see, for example, Forest [19]).

Recently, the non-local theory has been used to study fracture problems in piezoelectric materials [20–21]. In [22–25] the state of stress near the tip of a sharp line crack in an elastic plate subjected to uniform tension, in-plane shear and anti-plane shear were discussed. The field equations employed for the solution of these problems are those of the theory of non-local elasticity. The solutions gave finite stress at the crack tips, thus resolving a fundamental problem that has remained unsolved for over half a century. This enabled us to employ the maximum-stress hypothesis to deal with fracture problems and the composite-materials problem in a natural way. However, they were not exact and there is oscillatory stress near the crack tip [22]. The iteration error has a significant effect on the result [23–25], because the second Fredholm integral equation has a super-singularity integral kernel. To overcome the difficulty, the Schmidt method [26] will be used. As discussed in [27–32], the Schmidt method can be used to solve this kind of triple (dual) integral equation and the limit of the kernel does not tend to a constant. Recently, the same problems that were defined in [22–25] were tackled in [27–29] by means of the Schmidt method and the results are more accurate than those of Eringen [22-25]. In [30-32] the dynamic crack problems were investigated by use of non-local theory. To the best of authors' knowledge, analytical treatment of the transient problem of two I-cracks by using non-local theory has not been attempted.

For the above-mentioned reasons, the present paper deals with the dynamic problem of two collinear cracks in an elastic plate by use of non-local theory. The field equations of non-local elasticity theory are employed to formulate and solve this problem. To obtain the theoretical solution, a one-dimensional non-local kernel function is used instead of a two-dimensional kernel function for the anti-plane dynamic problem to obtain the stress occurring at the crack tips. To further simplify our analysis, the same assumptions as Nowinski's [18, 33] are made in this paper. Certainly, the assumption should be further investigated to satisfy the realistic condition. The Fourier-transform technology is applied and a mixed-boundary-value problem is formulated. Then a set of triple integral equations is solved by a new method,

namely, the Schmidt method [26]. In solving the equations, the crack-surface displacement is expanded in a series of Jacobi polynomials. The Schmidt method is used to obtain the solutions. This process is quite different from those adopted in Eringen's work [22–25] and overcome the mathematical difficulties experienced therein. The solution in this paper is more accurate and more appropriate than Eringen's. The solution, as expected, does not contain the stress singularity near the crack tips. The stress along the crack line depends not only on the crack length, but also on the lattice parameter and the circular frequency of the incident wave. However, the stress resulting from classical theory depends only on crack length and the circular frequency of the incident wave.

2. Basic equations of non-local elasticity

According to non-local theory, the stress at a point X in a body depends not only on the strain at point X, but also on those at all other points of the body. This observation is in accordance with atomic theory of lattice dynamics and experimental observation of phonon dispersion [34]. Basic equations of linear, homogeneous, isotropic, non-local elastic solids, with vanishing body force are

$$\tau_{kl,k} = \rho \ddot{u}_l, \tag{1}$$

$$\tau_{kl} = \int_V \alpha(\left|X' - X\right|)\sigma_{kl}(X', t)\,\mathrm{d}V(X'), \tag{2}$$

where

$$\sigma_{ij}(X', t) = \lambda u_{r,r}(X', t)\delta_{ij} + \mu[u_{i,j}(X', t) + u_{j,i}(X', t)], \quad (i, j \in (1, 3)), \tag{3}$$

where the only difference with classical theory is Equation (2); the stress $\tau_{kl}(X)$ at a point X depends on the strains $e_{kl}(X')$ at all points of the body. For homogeneous and isotropic solids there exist only two material constants; λ and μ are the Lame constants of classical elasticity, ρ is the mass density of the material. $\alpha(\left|X' - X\right|)$ is known as the influence function; it is a function of the distance $\left|X' - X\right|$. Expression (3) is the classical Hook's law. Substitution of Equation (3) in Equation (2) and use of the Green-Gauss theorem leads to

$$\int_V \alpha(\left|X' - X\right|)[(\lambda + \mu)u_{k,kl}(X', t) + \mu u_{l,kk}(X', t)]\mathrm{d}V(X')$$

$$- \int_{\partial V} \alpha(\left|X' - X\right|)\sigma_{kl}(X', t)\mathrm{d}a_k(X') = \rho \ddot{u}_l, (k, l \in (1, 3)) \tag{4}$$

In this paper, we only consider the perturbation field. The surface integral at infinity is zero.

3. The crack model

It is assumed that there are two collinear symmetric cracks of length $l = 1 - b$ along the x-axis with the distance between the two cracks being $2b$ (see Figure 1.). For the problem of two collinear symmetric cracks of arbitrary finite length $l' = c - b(c > b > 0)$, the solution can easily be obtained by a simple change in the numerical values of the present problem. In this paper, we only consider the two-dimensional plane strain problem. For the

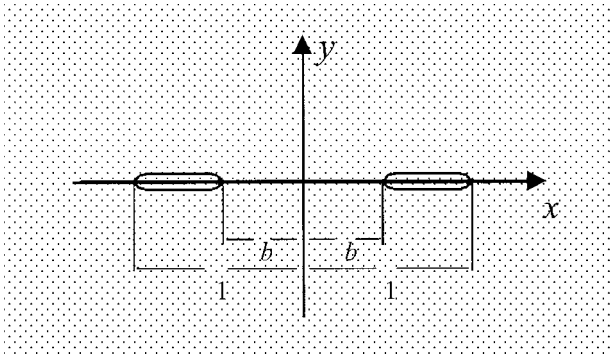

Figure 1. Incidence of a time-harmonic wave on two collinear symmetric cracks of the length $l = 1 - b$.

two-dimensional plane stress problem, λshould be replaced by $\lambda^* = 2\mu\lambda/(\lambda + 2\mu)$. Let ω be the circular frequency of the incident wave; $-\tau_0$ is the magnitude of the incident wave. In what follows, the time dependence term $\exp(-i\omega t)$ will be suppressed, which is a commonly used technique. It is further assumed that the two faces of the crack do not come in contact during vibrations as stated in Srivastava's paper [35]. When the cracks are subjected to harmonic elastic waves, as discussed in [35], the boundary conditions on the crack faces at $y = 0$ are (b is a dimensionless variable):

$$\tau_{yx}(x, 0, t) = 0, \quad v(x, 0, t) = 0, \quad |x| \geq 1, \quad |x| \leq b, \tag{5}$$

$$\tau_{yx}(x, 0, t) = 0, \quad \tau_{yy}(x, 0, t) = -\tau_0, \quad b < |x| < 1, \tag{6}$$

$$u(x, y, t) = v(x, y, t) = 0 \quad , \quad (x^2 + y^2)^{1/2} \to \infty, \tag{7}$$

In this paper, the wave is vertically incident and we only consider positive τ_0.

4. The triple integral equations

According to the boundary conditions, Equation (4) can be written as follow:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(|x' - x|, |y' - y|)[(\lambda + \mu)u_{k,kj}(x', y', t) + \mu u_{j,kk}(x', y', t)]\,dx'\,dy'$$

$$-2\{\int_{-1}^{-b} + \int_{b}^{1}\}\alpha(|x' - x|, |y|)[\sigma_{2j}(x', 0, t)]dx' = -\rho\omega^2 u_j, (i, j, k \in (1, 3)), \tag{8}$$

where $[\sigma_{2j}(x, 0, t)] = \sigma_{2j}(x, 0^+, t) - \sigma_{2j}(x, 0^-, t)$ is a jump across the crack.

From [5] and [7], we obtain:

$$\left[\sigma_{2j}(x, 0, t)\right] = 0 \text{ for all } x, \tag{9}$$

Define the Fourier transform by the equations

$$\bar{f}(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}\,dx, \tag{10}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s, \tag{11}$$

To solve the problem, the Fourier transform of Equation (8) with respect x can be given as follows:

$$\int_{-\infty}^{\infty} \bar{\alpha}(|s|, \left|y' - y\right|)[\mu\bar{u}_{,yy} - (\lambda + 2\mu)s^2\bar{u} - \mathrm{i}s(\lambda + \mu)\bar{v}_{,y}]\,\mathrm{d}y' = -\omega^2\rho\bar{u}, \tag{12}$$

$$\int_{-\infty}^{\infty} \bar{\alpha}(|s|, \left|y' - y\right|)[-\mathrm{i}s(\lambda + \mu)\bar{u}_{,y} + (\lambda + 2\mu)\bar{v}_{,yy} - s^2\mu\bar{v}]\,\mathrm{d}y' = -\omega^2\rho\bar{v}. \tag{13}$$

What now remains is to solve the integrodifferential equations (12–13) for the functions u and v. It is impossible to obtain a rigorous solution at the present stage. It seems obvious that in the solution of such a problem we encounter serious if not unsurmountable mathematical difficulties and will have to resort to an approximate procedure. In the given problem, according to Nowinski [18, 33], the appropriate numerical procedure seems to follow naturally from the hypothesis of the attenuating neighborhood underlying the theory of non-local continua. According to this hypothesis, the influence of the particle of the body, on the thermoelectric state at the particle under observation, diminishes fairly rapidly with increasing distance from the particle. In classical theory, the function that characterizes particle interactions is the Dirac delta function, since in this theory the actions are assumed to have zero range. In non-local theories the intermolecular forces may be represented by a variety of functions as long as their values decrease rapidly with distance. In the present study, as adequate functions it was decided to select the terms, $\delta_n(y' - y)$, $n = 1, 2, \ldots$, of the so-called δ-sequences. A δ-sequence, as is generally known, is (in the present case) a one-dimensional Dirac delta function, $\delta(y' - y)$. With respect to the terms of the delta sequence the following simplifying assumptions are adopted: (see [18, 33], Nowinski solved several non-local problems by using this kind of assumption):

(a) : For a sufficiently large j(as compared with the sphere of interactions of the particles), it is permissible to make the replacement

$$\int_{-j}^{j} f(y')\delta_n(y' - y)\mathrm{d}y' \approx \int_{-\infty}^{\infty} f(y')\delta(y' - y)\mathrm{d}y', \tag{14}$$

(b): As a consequence, the terms $\delta_n(y - y)$, with $n\rangle\rangle 1$, acquire the shifting property of the Dirac function,

$$\int_{-j}^{j} f(y')\delta_n(y' - y)\mathrm{d}y' \approx f(y), \tag{15}$$

The influence function is sought in a separable form. So, according to the above discussion, the non-local interaction in the y-direction can be ignored. In view of our assumptions, it can be given as

$$\bar{\alpha}(|s|, \left|y' - y\right|) = \bar{\alpha}_0(s)\delta_n(y' - y) \tag{16}$$

From Equations (12) and (13), we have

$$\bar{\alpha}_0(s)[\mu\bar{u}_{,yy} - (\lambda + 2\mu)s^2\bar{u} - \mathrm{i}s(\lambda + \mu)\bar{v}_{,y}] = -\omega^2\rho\bar{u}, \tag{17}$$

$$\bar{\alpha}_0(s)[-\mathrm{i}s(\lambda+\mu)\bar{u}_{,y}+(\lambda+2\mu)\bar{v}_{,yy}-s^2\mu\bar{v}]=-\omega^2\rho\bar{v}, \tag{18}$$

the solution of which does not present any difficulties; thus we have ($y \geq 0$)

$$\begin{aligned} u(x,y,t) &= -\frac{2}{\pi}\int_0^\infty sA_1(s)\sin(sx)\exp(-\gamma_1 y)\,\mathrm{d}s \\ &\quad -\frac{2}{\pi}\int_0^\infty \gamma_2 A_2(s)\sin(sx)\exp(-\gamma_2 y)\,\mathrm{d}s, \end{aligned} \tag{19}$$

$$\begin{aligned} v(x,y,t) &= -\frac{2}{\pi}\int_0^\infty \gamma_1 A_1(s)\cos(sx)\exp(-\gamma_1 y)\,\mathrm{d}s \\ &\quad -\frac{2}{\pi}\int_0^\infty sA_2(s)\cos(sx)\exp(-\gamma_2 y)\,\mathrm{d}s, \end{aligned} \tag{20}$$

$$\gamma_1^2 = s^2-\frac{\omega^2}{c_1^2\bar{\alpha}_0(s)}, \quad \gamma_2^2 = s^2-\frac{\omega^2}{c_2^2\bar{\alpha}_0(s)}, \quad c_1=\sqrt{\frac{\lambda+2\mu}{\rho}}, \quad c_2=\sqrt{\frac{\mu}{\rho}}.$$

where $A_1(s)$ and $A_2(s)$ are unknown functions to be determined by the boundary conditions.

Now, let the function $A(s)$ be defined such that the boundary condition $\tau_{xy}(x,0)=0$ is automatically satisfied:

$$A_1(s) = -\frac{1}{2\gamma_1}[s^2+\gamma_2^2]\bar{\alpha}_0(s)A(s), \quad A_2(s) = s\bar{\alpha}_0(s)A(s), \tag{21}$$

Because of symmetry, it suffices to consider the problem in the first quadrant only. The boundary conditions (5) and (6) can be applied to yield

$$\int_0^\infty A(s)\cos(sx)\mathrm{d}s = 0, \quad x \geq 1, x \leq b, \tag{22}$$

$$\int_0^\infty \bar{\alpha}_0^2(s)f(s)A(s)\cos(sx)\,\mathrm{d}s = \frac{\tau_0\pi}{2\mu}, \quad b < x < 1. \tag{23}$$

Equations (22) and (23) are the triple integral equations of this problem. In Equation (23), $f(s)$ is given as follows:

$$f(s) = \frac{1}{2\gamma_1}\{[s^2+\gamma_2^2]^2-4s^2\gamma_1\gamma_2\}. \tag{24}$$

To determine the unknown function $A(s)$, the triple integral equations (22–23) must be solved.

5. Solution of the triple integral equation

Since the only difference between classical and non-local equations is in the introduction of the function $\bar{\alpha}_0^2(s)$, it is logical to utilize the classical solution to convert the system (22–23)

into an integral equation of the second kind that is generally better behaved. As discussed in [25] and [33], we should take

$$\alpha_0 = \chi_0 \exp(-(\beta/a)^2(x' - x)^2) \quad \text{with} \quad \chi_0 = \tfrac{1}{\sqrt{\pi}}\beta/a, \tag{25}$$

where β is a constant and a is the lattice parameter.

So we obtain

$$\bar{\alpha}_0(s) = \exp\left(-\frac{(sa)^2}{(2\beta)^2}\right), \tag{26}$$

with $\bar{\alpha}_0(s) = 1$ for the limit $a \to 0$ (We consider the crystal as a lattice of regularly spaced sites with lattice parameter a), so that Equations (22–23) reduce to the well-known triple integral equations of the classical theory. The triple integral equations for the same problem in classical fracture theory can be transformed into a Fredholm integral equation of the second kind. However, the triple integral equations (22–23) can not be transformed into a Fredholm integral equation of the second kind, because $\bar{\alpha}_0^2(s) f(s)$ does not tend to a constant $C(C \neq 0)$ for $s \to \infty$. This can be explained as follows. In Eringen's paper [25], the second-kind Fredholm integral equation can be rewritten as follows

$$h(x) + \int_0^1 h(u)L(x, u)\,\mathrm{d}u = g(x),$$

where $g(x)$ is a known function, and $h(x)$ is an unknown function.

The kernel of the above second-kind Fredholm integral equation is divergent, which can be written:

$$L(x, u) = (xu)^{\frac{1}{2}} \int_0^\infty tk(\varepsilon t) J_0(xt) J_0(ut)\,\mathrm{d}t, \quad 0 \leq x, u \leq 1,$$

where $J_n(x)$ is the Bessel function of order n.

$$k(\varepsilon t) = -\Phi(\varepsilon t), \quad \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)\mathrm{d}t, \quad J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - \frac{1}{4}\pi) \text{ for } x \gg 0,$$

$$\lim_{t \to \infty} k(\varepsilon t) \neq 0 \text{ for } \varepsilon = \frac{a}{2\beta l} \neq 0, \text{ (l is the length of the crack).}$$

The limit of $tk(\varepsilon t) J_0(xt) J_0(ut)$ is not equal to zero for $t \to \infty$. So the kernel $L(x, u)$ in [25] is divergent. Therefore, the results in Eringen's paper are not appropriate. Of course, the triple integral equations can be considered as a single integral equation of the first kind with a discontinuous kernel [22]. It is well-known from the literature that integral equations of the first kind are generally ill-posed in the sense of Hadamard, *i.e.* small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. In this paper, the Schmidt method [26] is used to solve the triple integral equations (22) and (23). The displacement v was represented by the following series:

$$v = \sum_{n=0}^{\infty} a_n P_n^{(\frac{1}{2}, \frac{1}{2})} \left(\frac{x - \frac{1+b}{2}}{\frac{1-b}{2}}\right) \left(1 - \frac{\left(x - \frac{1+b}{2}\right)^2}{\left(\frac{1-b}{2}\right)^2}\right)^{\frac{1}{2}}, \quad \text{for } b < x \leq 1,\ y = 0, \tag{27}$$

$$v = 0, \quad \text{for } x > 1, x < b, y = 0, \tag{28}$$

where a_n are unknown coefficients to be determined and $P_n^{(1/2,1/2)}(x)$ is a Jacobi polynomial [36, p. 1035]. The Fourier transformation of Equation (27) is [37, p. 38, p. 94]:

$$-\frac{\omega^2}{2c_2^2}A(s) = \bar{v}(s,0,t) = \sum_{n=0}^{\infty} a_n B_n G_n(s) \frac{1}{s} J_{n+1}\left(s\frac{1-b}{2}\right), \tag{29}$$

$$B_n = 2\sqrt{\pi}\frac{\Gamma(n+1+\frac{1}{2})}{n!}, \quad G_n(s) = \begin{cases} (-1)^{\frac{n}{2}} \cos\left(s\frac{1+b}{2}\right), n = 0, 2, 4, 6, ... \\ (-1)^{\frac{n+1}{2}} \sin\left(s\frac{1+b}{2}\right), n = 1, 3, 5, 7, ... \end{cases}, \tag{30}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Equation (29) in Equations (22) and (23), we have that Equation (22) is automatically satisfied; Equation (23) reduces to the form for $b < x < 1$

$$\sum_{n=0}^{\infty} a_n B_n \int_0^{\infty} \bar{\alpha}_0^2(s) G_n(s) f(s) \frac{1}{s} J_{n+1}\left(s\frac{1-b}{2}\right) \cos(xs)\,\mathrm{d}s = -\frac{\tau_0 \omega^2 \pi}{4\mu c_2^2}, \tag{31}$$

For large s, almost all the integrands of Equation (31) decrease exponentially. Hence they can be evaluated numerically by the Filon method [38, pp. 105–123]. Thus Equation (31) can be solved for coefficients a_n by the Schmidt method [26]. For brevity, Equation (31) can be rewritten as

$$\sum_{n=0}^{\infty} a_n E_n(x) = U(x), \quad b < x < 1, \tag{32}$$

where $E_n(x)$ and $U(x)$ are known functions and a_n are unknown coefficients. A set of functions $P_n(x)$ which satisfy the orthogonality condition

$$\int_b^1 P_m(x) P_n(x) \mathrm{d}x = N_n \delta_{mn}, \quad N_n = \int_b^1 P_n^2(x)\,\mathrm{d}x, \tag{33}$$

can be constructed from the function $E_n(x)$ such that

$$P_n(x) = \sum_{i=0}^{n} \frac{M_{in}}{M_{nn}} E_i(x), \tag{34}$$

where M_{ij} is the cofactor of the element d_{ij} of D_n, which is defined as

$$D_n = \begin{bmatrix} d_{00}, d_{01}, d_{02}, \ldots, d_{0n} \\ d_{10}, d_{11}, d_{12}, \ldots, d_{1n} \\ d_{20}, d_{21}, d_{22}, \ldots, d_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ d_{n0}, d_{n1}, d_{n2}, \ldots, d_{nn} \end{bmatrix}, \quad d_{ij} = \int_b^1 E_i(x) E_j(x)\,\mathrm{d}x, \tag{35}$$

Table 1. Values of $\sum_{n=0}^{9} a_n E_n(x) \Big/ \frac{\pi\tau_0\omega^2}{4\mu c_2^2}$ and $U(x) \Big/ \frac{\pi\tau_0\omega^2}{4\mu c_2^2}$ for $b = 0{\cdot}1$, $a/(2\beta) = 0{\cdot}0005$, $\omega/c_2 = 0{\cdot}2$.

x	$\sum_{n=0}^{9} a_n E_n(x) \Big/ \frac{\pi\tau_0\omega^2}{4\mu c_2^2}$	$U(x) \Big/ \frac{\pi\tau_0\omega^2}{4\mu c_2^2}$
0·1	−1·0080	−1·0
0·2	−1·0047	−1·0
0·3	−1·0200	−1·0
0·4	−1·0005	−1·0
0·5	−1·0597	−1·0
0·6	−1·0095	−1·0
0·7	−1·0034	−1·0
0·8	−1·0056	−1·0
0·9	−1·0078	−1·0

Using Equations (32–35), we obtain

$$a_n = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \text{ with } q_j = \frac{1}{N_j} \int_b^1 U(x) P_j(x)\, \mathrm{d}x. \tag{36}$$

6. Numerical calculations

For a check of the accuracy, the values of $\sum_{n=0}^{9} a_n E_n(x)$ and $U(x)$ are given in Table 1 for $b = 0{\cdot}1$, $a/(2\beta) = 0{\cdot}0005$, $\omega/c_2 = 0{\cdot}2$. In Table 2 the values of the coefficients a_n are given for $b = 0{\cdot}1$, $a/(2\beta) = 0{\cdot}0005$, $\omega/c_2 = 0{\cdot}2$.

From [27–32, 39] and the above discussion, it can be seen that the Schmidt method works satisfactorily if the first ten terms of the infinite series of Equation (32) are retained. We can determine the entire dynamic stress field from the coefficients a_n. It is of importance in fracture mechanics to determine the dynamic stress τ_{yy} in the vicinity of the crack tips; τ_{yy} along the crack line can be expressed as

$$\tau_{yy} = \frac{4\mu c_2^2}{\omega^2 \pi} \sum_{n=0}^{\infty} a_n B_n \int_0^{\infty} \bar{\alpha}_0^2(s) G_n(s) f(s) \frac{1}{s} J_{n+1}(s\frac{1-b}{2}) \cos(sx)\, \mathrm{d}s. \tag{37}$$

From the above discussion, it can be deduced that the stress stays steady when the number of terms in (37) is increased; the semi-infinite integration

$$\int_0^{\infty} \bar{\alpha}_0^2(s) G_n(s) f(s) \frac{1}{s} J_{n+1}\left(s\frac{1-b}{2}\right) \cos(sx)\, \mathrm{d}s$$

and the series

$$\sum_{n=0}^{\infty} a_n B_n \int_0^{\infty} \bar{\alpha}_0^2(s) G_n(s) f(s) \frac{1}{s} J_{n+1}\left(s\frac{1-b}{2}\right) \cos(sx)\, \mathrm{d}s$$

Table 2. Values of $a_n \Big/ \frac{\pi\tau_0\omega^2}{4\mu c_2^2}$ for $a/(2\beta) = 0.0005$, $b = 0.1$, $\omega = 0.2$

n	$a_n/(\pi\tau_0\omega^2/4\mu c_2^2)$	
	Real part	Imaginary part
0	$-0{\cdot}350516 \times 10$	$-0{\cdot}150300 \times 10^{-1}$
1	$0{\cdot}158543 \times 10^{0}$	$0{\cdot}699877 \times 10^{-3}$
2	$-0{\cdot}760875 \times 10^{-1}$	$-0{\cdot}323144 \times 10^{-4}$
3	$0{\cdot}395745 \times 10^{-2}$	$0{\cdot}170345 \times 10^{-4}$
4	$-0{\cdot}314711 \times 10^{-2}$	$-0{\cdot}135059 \times 10^{-5}$
5	$0{\cdot}266151 \times 10^{-3}$	$0{\cdot}114774 \times 10^{-5}$
6	$-0{\cdot}257682 \times 10^{-4}$	$-0{\cdot}110740 \times 10^{-6}$
7	$0{\cdot}236005 \times 10^{-5}$	$0{\cdot}101800 \times 10^{-7}$
8	$-0{\cdot}177895 \times 10^{-5}$	$-0{\cdot}765896 \times 10^{-7}$
9	$0{\cdot}160583 \times 10^{-6}$	$0{\cdot}692765 \times 10^{-8}$

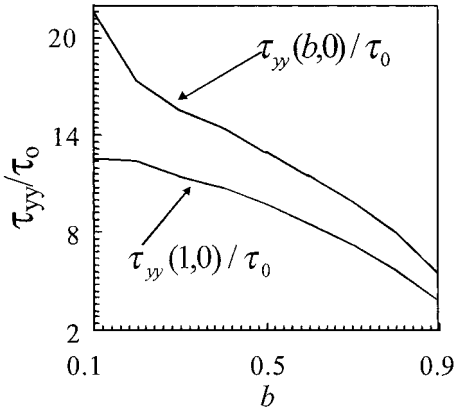

Figure 2. The variation of the stress at the crack tips for $a/(2\beta) = 0{\cdot}0005$, $\omega/c_2 = 0{\cdot}2$.

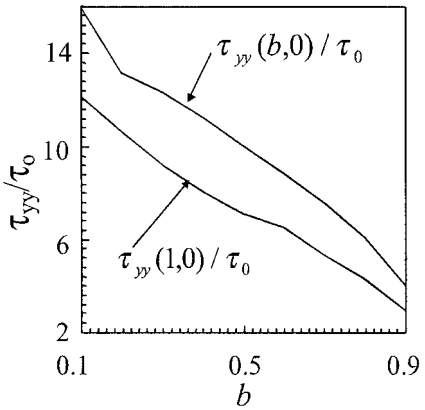

Figure 3. The variation of the stress at the crack tips for $a/(2\beta) = 0{\cdot}0008$, $\omega/c_2 = 1{\cdot}0$.

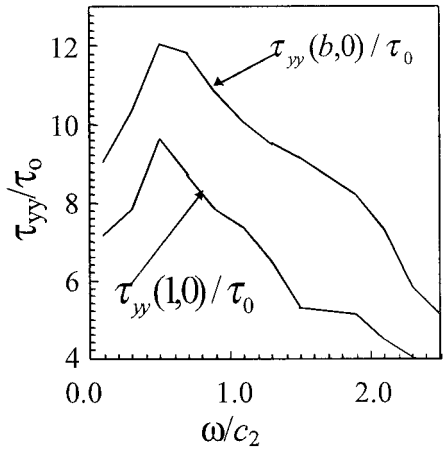

Figure 4. The variation of the stress at the crack tips for $a/(2\beta) = 0{\cdot}0008$, $b = 0{\cdot}4$.

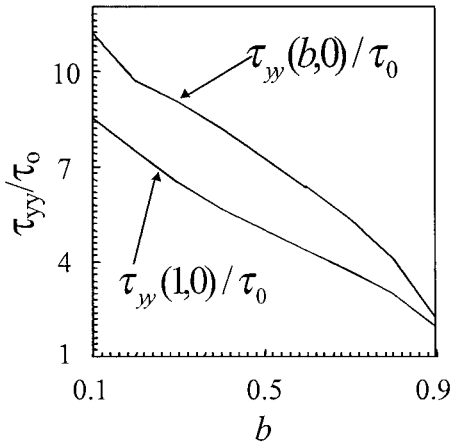

Figure 5. The variation of the stress at the crack tips for $a/(2\beta) = 0{\cdot}0015$, $\omega/c_2 = 1{\cdot}0$.

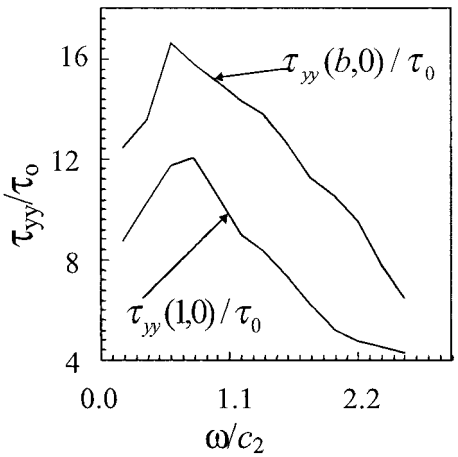

Figure 6. The variation of the stress along the crack line for $a/(2\beta) = 0{\cdot}0005$, $b = 0{\cdot}4$.

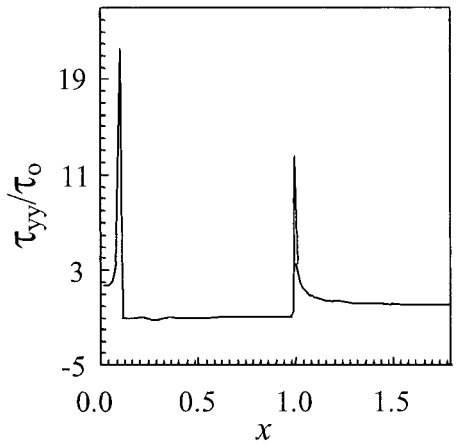

Figure 7. The variation of the stress along the crack line for $a/(2\beta) = 0{\cdot}0005$, $b = 0{\cdot}1$, $\omega/c_2 = 0{\cdot}2$.

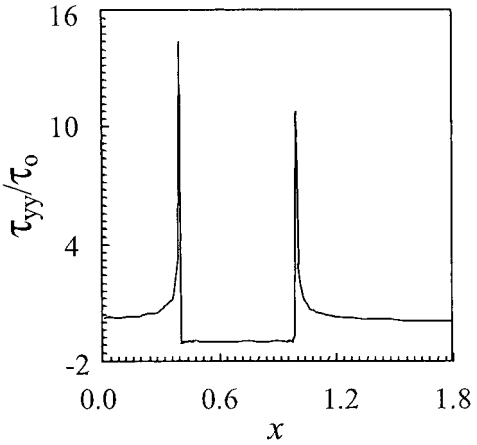

Figure 8. The variation of the stress along the crack line for $a/(2\beta) = 0{\cdot}0005$, $b = 0{\cdot}4$, $\omega/c_2 = 0{\cdot}2$.

in Equation (37) are convergent for any variable x with $a \neq 0$. Therefore the stress gives finite values all along the crack line. Contrary to the classical-theory solution, it is found that no stress singularity is present at the crack tip. For $a = 0$ at $x = 1, b$, we have classical stress singularity. At $b < x < 1$, τ_{yy}/τ_0 is very close to unity, and for $x > 1$, τ_{yy}/τ_0 possesses finite values diminishing from a maximum value at $x = 1$ to zero at $x = \infty$. Since $a/[2\beta(1-b)] > 1/100$ represents a crack length of less than 100 atomic distances as stated by Eringen [24], and for such tiny cracks other serious questions arise regarding the interatomic arrangements and force laws, we do not pursue solutions for such small crack sizes. The dynamic stress is computed numerically for the Lame constants $\lambda = 98 \times 10^9 (\mathrm{N/m^2})$, $\mu = 77 \times 10^9 (\mathrm{N/m^2})$, $\rho = 7{\cdot}7 \times 10^3 (\mathrm{kg/m^3})$. The semi-infinite numerical integrals are evaluated easily by the Filon and Simpson methods because of the rapid diminution of the integrands.

7. Discussion

The aim of the present paper was to study the application of the non-local theory in fracture mechanics. The other aim of the present paper was to show that the Schmidt method can be

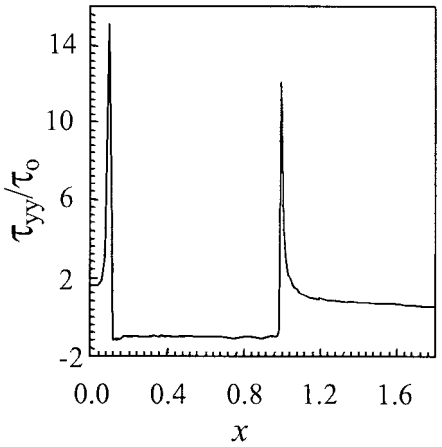

Figure 9. The variation of the stress along the crack line for $a/(2\beta) = 0{\cdot}0008$, $b = 0{\cdot}1$, $\omega/c_2 = 1{\cdot}0$.

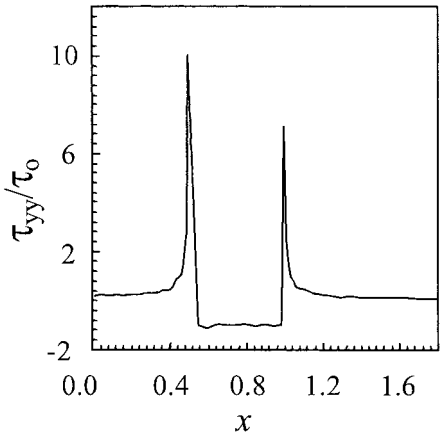

Figure 10. The variation of the stress along the crack line for $a/(2\beta) = 0{\cdot}0008$, $b = 0{\cdot}5$, $\omega/c_2 = 1{\cdot}0$.

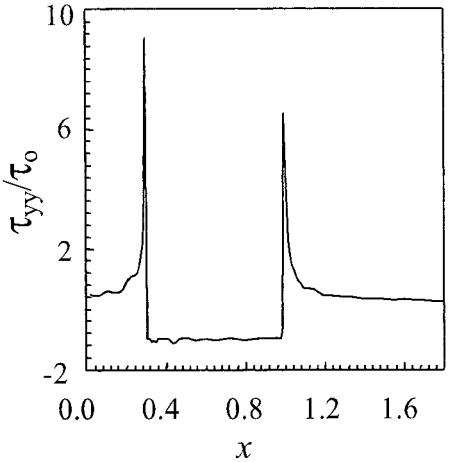

Figure 11. The variation of the stress along the crack line for for $a/(2\beta) = 0{\cdot}0015$, $b = 0{\cdot}3$, $\omega/c_2 = 1{\cdot}0$.

used to solve this kind of the triple (dual) integral equation and that the limit of the kernel is not a constant. This method is more exact and more appropriate than that put forward by Eringen for solving this kind of problem. Contrary to the classical-theory solution, it was found that no stress singularity is present at the crack tip and the stress is finite there. Furthermore, the effects of the geometry of the interacting cracks, the frequency of the incident wave and the lattice parameter upon the dynamic stress field of the crack were examined, *i.e.* the dynamic perturbation stress field depends on the geometry of the cracks, the frequency of the incident wave and the lattice parameter. So the perturbation stress field can be obtained by the geometry of the cracks, the frequency of the incident wave and the lattice parameter from the results in this paper. In this paper, we only made an attempt to relate our formulation to a problem in a lattice structure. However, there are many problems that should be investigated in future work on non-local theory. For example, the choice of the influence function α should be further studied to satisfy the realistic condition, the practical value of the maximum stress near the crack tips should be measured by experiments, and so on. However, the application of non-local theory is very limited since the basic equations of non-local theory are complex. The results are plotted in Figures 2–11. From the results the following observations can be made:

(i) The method used in this paper can overcome the mathematical difficulties that occur in Eringen's papers [22–25], *i.e.* the Schmidt method can be used to solve this kind of the triple (dual) integral equation where the limit of the kernel is not a constant. The results are more accurate than those of Eringen. The method is more appropriate than Eringen's.

(ii) The maximum stress does not occur at the crack tip, but slightly away from it. This phenomenon was thoroughly substantiated by Eringen [40]. The maximum stress is finite. The distance between the crack tip and the maximum stress point is very small. This distance depends on the lattice parameter, the crack length and the circular frequency of the incident wave. Contrary to the classical-elasticity solution, it is found that no stress singularity is present at the crack tip, and also that the present results converge to the classical ones for positions far away from the crack tip as shown in Figures 7–11. This enabled us to employ the maximum-stress hypothesis to deal with the engineering fracture problem in a natural way. The maximum-stress fracture criterion is more exact than 'the stress intensity factor' criterion of the classical theory.

(iii) The normal stress at the crack tip becomes infinite when the atomic distance $a \to 0$. This is the classical continuum limit of square-root singularity.

(iv) For $a/\beta =$ constant, *i.e.* the atomic distance does not change, the value of the dynamic stress concentrations (at the crack tip) increases with an increase of the crack length. Experiments indicate that materials with smaller cracks are more resistant to fracture than those with larger cracks.

(v) The significance of this result is that the fracture criteria are unified at both macroscopic and microscopic scales.

(vi) The tip stress at the left tip is greater than that at the right tip for the right crack. The stress on the crack line decreases with increasing distance between two cracks as shown in Figure 2, Figure 3 and Figure 5.

(vii) The present solution will revert to the classical one for $\alpha(|X' - X|) = \delta(|X' - X|)$.

(viii) The dynamic stress at the crack tips tends to increase with frequency, reaches a peak and then decreases in magnitude. It can be shown that the stresses at the crack tips show a maximum value near a certain frequency. So the stress field can reach its minimum value by changing the frequency of the incident wave, the lattice parameter and the length of the crack.

(ix) The results in Figures 2–11 reflect the influence of the lattice parameter on the perturbation stress field near the crack tips for various materials, *i.e.* the perturbation stress field increases for a decreasing value of the lattice parameter. So we can predict the influence of the crack on the stress distribution of the materials according to the lattice parameter.

Acknowledgements

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